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FRACTAL DIMENSION, PRIMES, AND THE PERSISTENCE OF MEMORY

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Many sequences from number theory, such as the primes, are defined by recursive procedures, often leading to complex local behavior, but also to graphical similarity on different scales — a property that can be analyzed by fractal dimension. This paper computes sample fractal dimensions from the graphs of some number-theoretic functions. It argues for the usefulness of empirical fractal dimension as a distinguishing characteristic of the graph. Also, it notes a remarkable similarity between two apparently unrelated sequences: the *persistence* of a number, and the memory of a prime. This similarity is quantified using fractal dimension.

Keywords: Empirical fractal dimension; number-theoretic functions; persistence of a number; memory of a prime; oscillator sequences.

1. Introduction

The graphs of number-theoretic functions, although not fractals in the proper sense, can exhibit striking similarity on various scales. For example, consider the sequence $a(n)$ defined recursively by: $a(1) = 1$; $a(n) = 1 - [p(n-1)/p(n)]a(n-1)$ if $n > 1$, where $p(n)$ denotes the nth prime. $a(n)$ is called the *oscillator sequence* of $p(n)$. (Generally, if p denotes a sequence such that $p(n)$ is never equal to 0, then the oscillator sequence $a(n)$ of $p(n)$ is defined by the above recursive procedure.) In Fig. 1, the graph of $a(n)$ from $n = 1$ to 10^5 exhibits an unexpected fractal structure: parts of the graph look like scaled-down copies of other parts.

Fractals are often studied with the aid of *fractal dimension*. Fractal dimension is a measure of the "convolutedness" or "degree of meandering" of the fractal set; for more information on fractal dimension, see Refs. 1 or 2. This paper examines sample fractal dimensions of some arithmetical functions from both classical and recreational number theory.

The paper [5] presents a fast and simple way of estimating the fractal dimension of a wave-form (in the plane). The term *wave-form* refers to "the shape of a wave, usually drawn as instantaneous values of a periodic quantity *versus* time" [5]. In

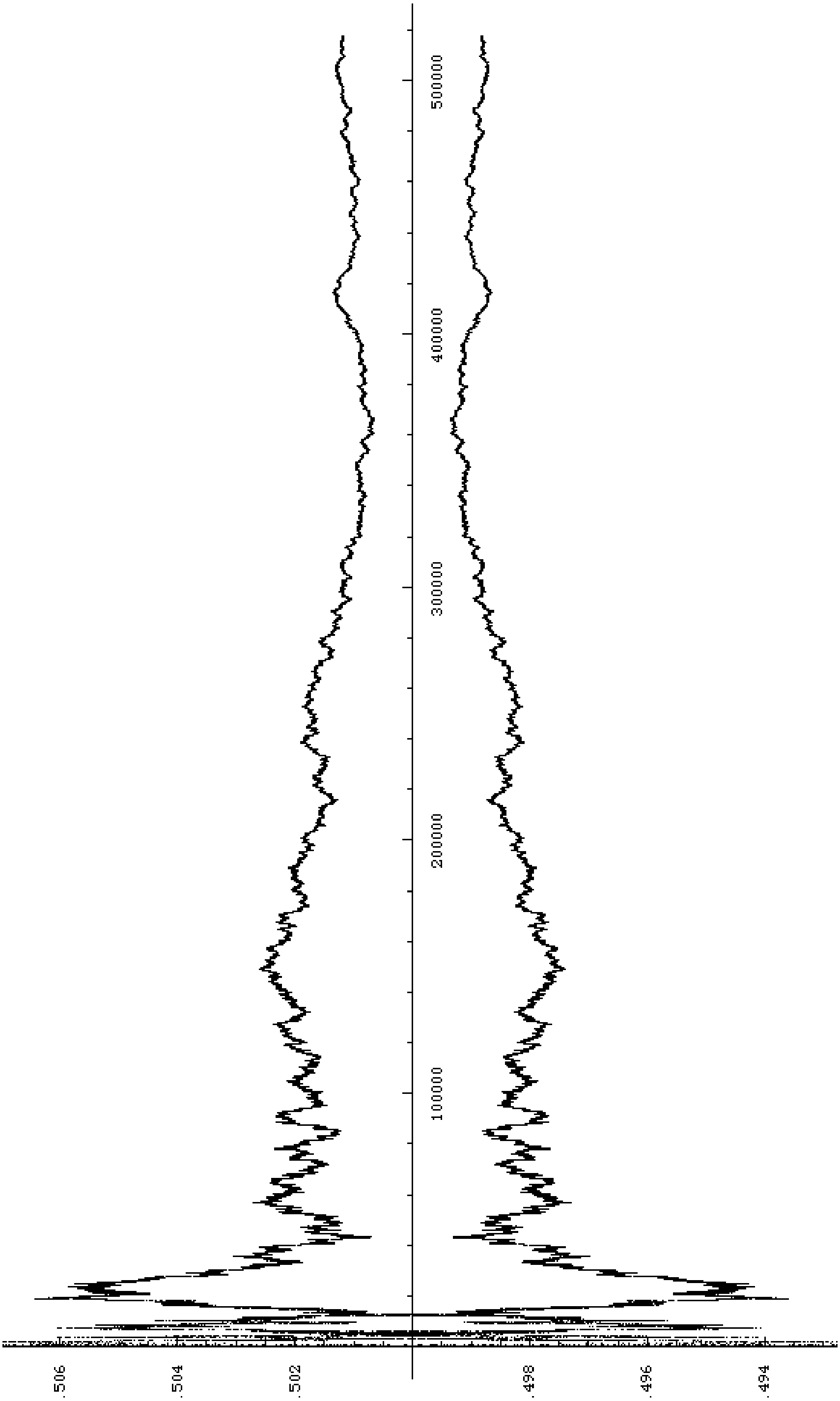

Fig. 1. Graph of $a(n)$ from $n = 1$ to 10^5.

practice, a wave-form is represented by a finite sample of N points $W = \{(x_i, y_i) : i = 1, \ldots, N\}$ evenly spaced along the x-axis.

To estimate the fractal dimension D of the wave-form $W = \{(x_i, y_i) : i = 1, \ldots, N\}$, W is first normalized using the formulas

$$x_i^* = \frac{x_i}{x_{\max}}, \qquad y_i^* = \frac{y_i - y_{\max}}{y_{\max} - y_{\min}} \tag{1}$$

to obtain $W^* = \{(x_i^*, y_i^*) : i = 1, \ldots, N\}$; here $x_{\max}$, $y_{\max}$ denote the maximum values of the numbers x_i, y_i, respectively, and $y_{\min}$ denotes the minimum of the numbers y_i. The fractal dimension D can be estimated using the formula

$$D = 1 + \frac{\ln(L)}{\ln(2N^*)}, \tag{2}$$

where L is the length of the normalized wave-form W^*, and $N^* = N - 1$. L is easily calculated by repeated application of the distance formula on (x_i^*, y_i^*) and (x_{i+1}^*, y_{i+1}^*) for $i = 1, 2, \ldots, N^*$, that is

$$L = \sum_{i=1}^{N^*} \sqrt{(x_{i+1}^* - x_i^*)^2 + (y_{i+1}^* - y_i^*)^2}\,. \tag{3}$$

In this paper, graphs of some arithmetical functions will be treated *as if they were waveforms*, and their fractal dimensions estimated using Eq. (2). Strictly speaking, the fractal dimensions of the graphs cannot be defined rigorously as limits (as done in Ref. 1) because of the non-compactness of these sets. What will be obtained are "sample fractal dimensions" — a statistical, empirically based concept. Although the graphs are neither waveforms nor classical fractals, they are generated by rules that are recursive and independent of scale. (For example, the primes are generated using the sieve of Eratosthenes.) Intuitively, such rules induce similarity — at the very least, statistical similarity — on different scales and make the notion of sample or empirical fractal dimension meaningful. The overall consistency of the estimates appearing below points to the existence of approximately well-defined fractal dimensions, and hence, supports the applicability of the fractal dimension notion to the graphs.

If $f(n)$ defines an arithmetical function, and a, b are integers with $a < b$, then the *sample of* $f(n)$ *from* $n = a$ *to* b is defined as the set of points $W = \{(a, f(a)), (a+1, f(a+1)), \ldots, (b, f(b))\}$. Note that only samples with points evenly spaced by 1 along the abscissa are considered here. For example, if $p(n) =$ the nth prime, then the sample of $p(n)$ from $n = 1$ to 4 is $W = \{(1, 2), (2, 3), (3, 5), (4, 7)\}$. It is clear that Eq. (2) yields a value of D when applied to a sample.

2. The Joy of Dimensioning

This section presents the results of some fractal dimension calculations on various samples of arithmetical functions. More precisely, the fractal dimension is estimated using Eq.(2) on adjacent samples of equal size, and the values are averaged to yield

Table 1. Estimated fractal dimensions of samples in range $[a, b]$.

Range of n	$p(n)$	$\varphi(n)$	$\tau(n)$	$\sigma(n)$
$[1, 1000]$	1.05207	1.69979	1.70217	1.64791
$[1000, 2000]$	1.01965	1.76963	1.70662	1.71397
$[2000, 3000]$	1.01021	1.78924	1.69172	1.73423
$[3000, 4000]$	1.00632	1.79913	1.69912	1.73465
$[4000, 5000]$	1.00416	1.80498	1.70404	1.75305
$[5000, 6000]$	1.00303	1.80708	1.67594	1.74926
$[6000, 7000]$	1.00225	1.80878	1.68878	1.74923
$[7000, 8000]$	1.00176	1.81019	1.6732	1.74426
$[8000, 9000]$	1.00144	1.81075	1.68417	1.75418
$[9000, 10000]$	1.00118	1.81221	1.67717	1.7558
Mean	1.01021	1.79118	1.69029	1.73365
Standard Deviation	0.0157687	0.0346781	0.012413	0.0327065

a representative dimension. While no attempt is made to prove convergence of the different estimates, their consistently small variance strongly suggests that the curve possesses some fractal structure, and that the derived (average) dimension captures the "degree of meandering" of the entire curve.

The numerical results are presented in Table 1. This table lists, for some well-known arithmetical functions $f(n)$, the estimated fractal dimensions of samples corresponding to n in the closed intervals $[1, 2000]$, $[1000, 2000], \ldots, [9000, 10000]$. Although the common sample size is 1000, calculations done with other large sample sizes indicate similar results.

For example, in the row containing $[1, 1000]$, one reads that the (empirical) fractal dimension of the sample of $p(n) =$ the nth prime from $n = 1$ to 1000 is about 1.05207. In the same row, one reads that the fractal dimension of the sample of $\varphi(n)$ from $n = 1$ to 1000 is about 1.69979, where $\varphi(n)$ is Euler's totient function giving the number of positive integers less than n and co-prime to n. The fractal dimensions for the samples of $\tau(n)$ (the number of divisors of n) and $\sigma(n)$ (the sum of the divisors of n) from $n = 1$ to 1000 are about 1.70217, 1.64791, respectively. In the next row, one reads that the fractal dimension of the sample of $p(n)$ from $n = 1000$ to 2000 is about 1.01965, and so on.

In the column under $p(n)$, the fractal dimension estimates of samples for $n = 1$ to 1000, 1000 to $2000, \ldots, 9000$ to 10000 appear along with their mean and standard deviation. The same can be said for the other arithmetical functions. What is striking about the results is their small standard deviation, or equivalently, their high consistency throughout the different intervals. The estimates for $p(n)$, for example, have a mean of 1.01021 and a standard deviation of only 0.0157687. The closeness of the $p(n)$-mean to 1, the fractal dimension of a straight-line segment, should be expected, since the Prime Number Theorem guarantees the regular large-scale behavior of primes (for example, $p(n) \sim n \ln(n)$; see Ref. 3, p. 10). Hence, the graph of $p(n)$ should be about as "convoluted" as a straight-line segment.

The small standard deviations suggest that fractal dimension is intrinsic to the arithmetical functions considered in Table 1, rather than just a measure with only local validity. A potential application of this observation would be in the problem of recognizing whether a set S of consecutive points is contained in the graph of an arithmetical function f — a central problem for sequence databases such as in Ref. 7. Suppose that the fractal dimension estimates of samples of f have low standard deviation, so that their mean D can represent them. If the estimated fractal dimension of S is far from D, then there is good reason to believe that S is not contained in the graph of f. However, if the fractal dimensions are approximately equal, it can only be concluded that the "convolutedness" or "degree of meandering" of the two graphs are similar, as the next sections will show.

3. Oscillator Sequences

Of course, not all graphs have consistent sample fractal dimensions; and even among those that do, the consistency is often not immediately apparent. The graph of the oscillator sequence $a(n)$ of $p(n)$ defined in Sec. 1 has sample fractional dimensions that at first vary considerably, then appear to stabilize at around 1.9. The sample fractal dimensions of $a(n)$ from $n = 1$ to 1000, 1000 to 2000, 2000 to 3000, $\ldots$, 29000 to 30000 are respectively:

$$\begin{array}{l} 1.48932,\ 1.84434,\ 1.78545,\ 1.81604,\ 1.73604, \\ 1.78899,\ 1.85303,\ 1.87271,\ 1.87794,\ 1.83885, \\ 1.85961,\ 1.86793,\ 1.78133,\ 1.84995,\ 1.84795, \\ 1.87404,\ 1.89203,\ 1.88279,\ 1.88939,\ 1.89757, \\ 1.88931,\ 1.89954,\ 1.90197,\ 1.90077,\ 1.89752, \\ 1.89859,\ 1.89699,\ 1.89727,\ 1.89545,\ 1.89758\,. \end{array}$$

Incidentally, it is an open problem whether $a(n) \to 1/2$ or diverges.

Many oscillator sequences give sample dimensions close to 2, and so are nearly two-dimensional ("space-filling," like Peano's curve). For example, if $s(n)$ is the constant sequence mapping each positive integer to 1, then the oscillator sequence $a(n)$ oscillates between the values 0 and 1, and is of course, periodic and divergent. Computing the sample fractal dimensions from $n = 1$ to 1000, 1000 to 2000, and so on, gives basically a constant estimate of about 1.9. Similarly, the sequence $s(n) = n$ has an oscillator sequence with sample fractal dimensions that settle to about 1.9. On the other hand, the oscillator sequence of $s(n) = n^2$ has sample fractal dimensions that are very nearly $= 1$. The reader can verify these estimates as an exercise.

4. Persistence

N. Sloane first defined the *persistence* of a number in Ref. 8 as the number of times one needs to multiply the digits together before reaching a single digit. The persistence of n is denoted by $\mathrm{pers}(n)$. For example, $\mathrm{pers}(679) = 5$, as the following chain of digit products shows: $679 \to 378 \to 168 \to 48 \to 32 \to 6$ ($6 \times 7 \times 9 = 378$, etc.).

Sloane conjectured that the persistence of a number does not exceed a certain upper bound C. (Probably, $C = 11$.)

5. The Memory of a Prime

We introduced the notion of *memory of a prime* in A079066 of Ref. 7. The memory of $p(n)$ is defined to be the number of previous primes contained as sub-strings in $p(n)$. The sequence $\mu(n) = \text{memory}(p(n))$ begins

$$0, 0, 0, 0, 0, 1, 1, 0, 2, 1, 1, 2, 0, 1, 1, 2, 1, 0, 1, 1, 2, 1, 1, 0, 1, 0, 1, 1, 0, 3, \ldots.$$

For example, $\mu(9) = \text{memory}(p(9)) = \text{memory}(23) = 2$ since the smaller primes 2 and 3 are contained as sub-strings in 23. The least prime with memory $= 3$ is 113, which contains the primes 3, 11, and 13 as sub-strings.

6. The Persistence of Memory

At first glance, the sequences $\rho(n) = \text{pers}(n)$ and $\mu(n) = \text{memory}(p(n))$ appear to be totally unrelated, except for the occurrence of the words "persistence" and "memory" in the title of a famous Salvador Dalí painting. However, the graphs of these functions exhibit an intriguing similarity. In Figs. 2 and 3, these functions are graphed for $n = 1200$ to 1400; the similarity is also present in plots using

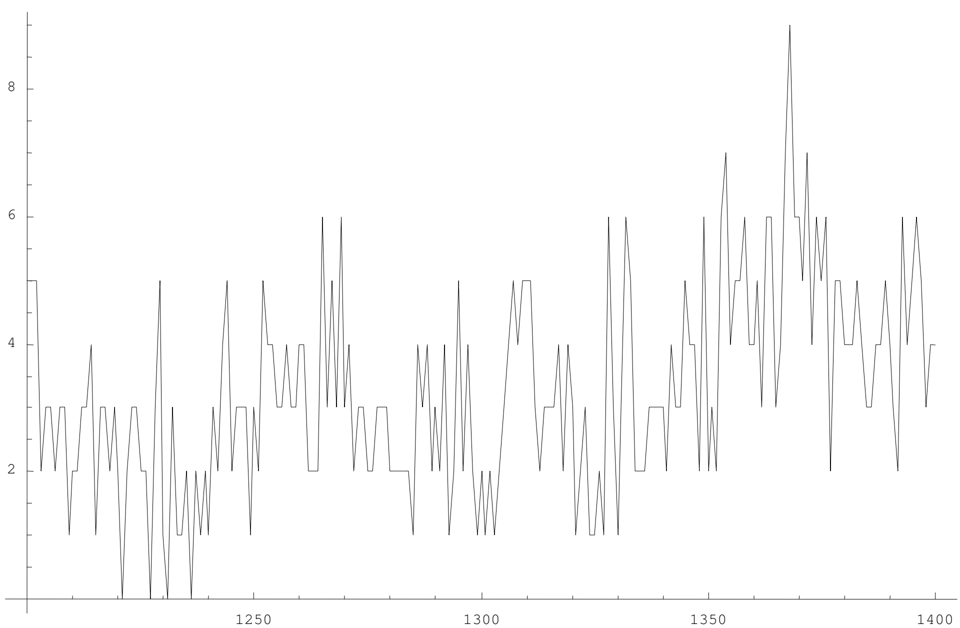

Fig. 2. Graph of $\mu(n) = \text{memory}(p(n))$ from $n = 1200$ to 1400.

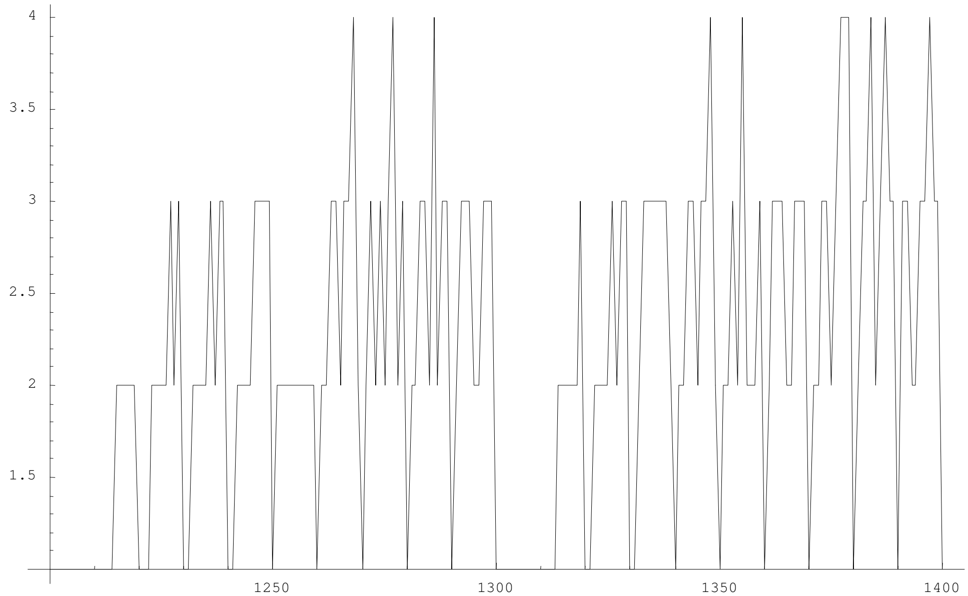

Fig. 3. Graph of $\rho(n) = \text{pers}(n)$ from $n = 1200$ to 1400.

other intervals. Note the common shape of the peaks, troughs and plateaus in both graphs.

However, one may be hard pressed to express exactly where the pictorial similarity lies. The curves share none of the classical graphical descriptors (for example, intercepts, maxima and minima) in common. Nor is one a scaled-down copy of the other. On closer inspection, the similarity appears to be in the way the curves "meander"... the fractal dimension comes to mind!

In Table 2, the columns corresponding to ρ and μ show the remarkable approximate equality of the (mean) fractal dimensions of these two curves (about 1.67118

Table 2. Estimated fractal dimensions of samples in range $[a, b]$.

Range of n	$\rho(n) = \text{pers}(n)$	$\mu(n) = \text{memory}(p(n))$	$\text{pers}(p(n))$
$[1, 1000]$	1.6463	1.68512	1.66399
$[1000, 2000]$	1.66668	1.66678	1.6454
$[2000, 3000]$	1.67543	1.66443	1.65309
$[3000, 4000]$	1.69177	1.66564	1.66681
$[4000, 5000]$	1.69073	1.67143	1.65807
$[5000, 6000]$	1.6118	1.66339	1.56324
$[6000, 7000]$	1.67037	1.67665	1.66715
$[7000, 8000]$	1.68642	1.67867	1.66443
$[8000, 9000]$	1.67068	1.67449	1.65625
$[9000, 10000]$	1.70162	1.67758	1.61035
Mean	1.67118	1.67242	1.64488
Standard Deviation	0.026139	0.00725178	0.0332543

for ρ and 1.67242 for μ). This probably accounts for the "hard to express" similarity. Fractal dimension estimates for samples of pers($p(n)$) also appear in Table 2, but the (mean) fractal dimension appears to be somewhat different from those of ρ and μ.

The approximate equality of the fractal dimensions of ρ and μ hint at a hidden connection in the rules used to generate the sequences. The exact nature of this connection is an open problem.

7. Mathematica Code

Here is the *Mathematica* code used in this paper to estimate fractal dimension:

```
(*MODULE TO ESTIMATE THE FRACTAL DIMENSION OF A SET {{x_i, y_i}} OF
DATA POINTS*)
fd[lst_] := Module[{clen, l, mat, i, abscc, ordn, xmax, ymax, ymin},
      clen = 0;
      l = Length[lst];
      mat = Transpose[lst];
      abscc = mat[[1]];
      ordn = mat[[2]];
      xmax = Max[abscc];
      ymax = Max[ordn];
      ymin = Min[ordn];
      diff = ymax - ymin;
      abscc = (1/xmax)*abscc;
      ordn = (1/diff)*(ordn - ymax);
      For[i = 2, i <= l, i++,
        clen =
          clen + Sqrt[(abscc[[i]] - abscc[[i - 1]])^2 + (ordn[[i]] -
                      ordn[[i - 1]])^2]];
      d = 1 + (Log[clen]/Log[2 (l - 1)]);
      N[d]];
```

To compute the fractal dimension of the sample of $p(n)$ from $n = 1$ to 1000:

```
fd[Table[{i,Prime[i]},{i,1,1000}]
```

To compute the persistence of a number:

```
pers[n_] := Module[{i, d, l},
      i = 0;
 d = IntegerDigits[n];
   l = Length[d];
 While[l > 1,
   p = Product[d[[i]], {i, 1, l}];
   d = IntegerDigits[p];
   l = Length[d];
    i = i + 1];
  i];
```

To compute the memories of the first 1000 primes:

```
      lb = 1;
      ub = 1000;
      tprime = Table[ToString[Prime[i]], {i, 1, ub}];
      a = {};
      For[i = lb, i <= ub, i++,
        m = 0;
        For[j = 1, j < i, j++,
         If[Length[StringPosition[tprime[[i]], tprime[[j]]]] > 0, m = m
+ 1]];
        a = Append[a, {i, m}]];
a
```

To plot the oscillator sequence of $p(n)$ and estimate its fractal dimension:

```
t = {{1,1}};
gt = 1;
For[i = 2, i <= 10^3, i++,
   gt = 1 - (Prime[i - 1]/Prime[i]) gt;
   t = Append[t, {i,gt}]];
fd[t]

ListPlot[t]
```

Acknowledgment

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References

[1] Barnsley, M., *Fractals Everywhere* (Academic Press, San Diego, 1988).
[2] Crownover, C., *An Introduction to Fractals and Chaos* (Jones and Bartlett, Boston, 1995).
[3] Hardy, G. and Wright, E., *Introduction to the Theory of Numbers*, 5th edn. (Oxford University Press, 1983).
[4] Lauwerier, H., *Fractals* (Princeton University Press, 1987).
[5] Sevcik, C., "A procedure to estimate the fractal dimension of waveforms," appeared in *Complexity International*, Vol. 5 (1998). The article is also available at the URL: http://www.csu.edu.au/ci/vol05/sevcik/sevcik.html
[6] Schroeder, M., *Number Theory in Science and Communication*, 3rd edn. (Springer-Verlag, 1997).
[7] Sloane, N., "The online encyclopedia of integer sequences," at the URL: http://www.research.att.com/∼njas/sequences/Seis.html
[8] Sloane, N., The persistence of a number, *J. Recreational Math.* **6**(2), 97–98 (1973).