The $3x + 1$ Fractal

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ABSTRACT. The $3x + 1$ sequence (also known as the Collatz sequence) is generalized to the complex function F: $\mathbf{C} \rightarrow \mathbf{C}$ defined by $F(z) = z/2$ if ceiling($|z|$) is even; otherwise $= 3z + 1$. Extensions of the well-known $3x + 1$ conjecture are considered. A striking fractal in the complex plane $\mathbf{C}$ is constructed from iteration of F and a density plot of the resulting modulus. Also, some variants of F are studied.

Keywords. $3x + 1$ problem. Collatz function. Complex dynamics. Fractal graphics.

1. Introduction

The *3x + 1 problem*, also known as the *Collatz problem*, is one of the most famous unsolved problems in number theory. It consists in proving or disproving the validity of the *3x + 1 conjecture*, which can be stated as follows. Consider the *Collatz function* defined by $f(n) = n/2$ if n is even, and $= 3n + 1$ if n is odd. It has been verified up to large positive integer n (around $n = 10^{15}$) that the sequence

$$n, f(n), f(f(n)), ...,$$

also called the *trajectory at n*, eventually reaches 1. For example, the corresponding sequence for $n = 12$ is: 12, 6, 3, 10, 5, 16, 8, 4, 2, 1, …. The $3x + 1$ conjecture states that, for any positive integer n, the trajectory at n reaches 1.

Although it is widely believed to be true, the $3x + 1$ conjecture has resisted numerous attempts at resolution. At the time of writing, a solution is nowhere in sight, and progress has been mostly in probabilistic arguments. The problem has often been labeled “intractable”. Indeed, Paul Erdös has remarked: "Mathematics is not yet ready for such problems." The reader is referred to [1] for additional background.

In this paper, more darkness will be thrown over the mystery by considering the $3x + 1$ constructions in the context of the complex plane **C**. It will be seen that intriguing generalizations of these can be made in the new extended context. On the positive side, a remarkable fractal resulting from iteration of the complex Collatz function will be unveiled.

In what follows, z represents a complex number and n represents a positive integer, unless otherwise noted.

2. In the Realm of (More) Complexity

The function F: $\boldsymbol{C} \to \boldsymbol{C}$ defined by

$F(z) = z/2$ if ceiling($|z|$) is even

$= 3z + 1$, otherwise

is called the *complex Collatz function*. Here $|z| = [\mathrm{Re}(z)^2 + \mathrm{Im}(z)^2]^{1/2}$ denotes the modulus of z, and ceiling(x) represents the least integer $\geq x$, where x is a real number. The *trajectory of F at z* is defined similarly as for f. For example, $F(3 + 4\,i) = 3(3 + 4\,i) + 1 = 10 + 12\,i$ since ceiling($|3 + 4\,i|$) = 5 is odd. On the other hand, $F(2 + 3\,i) = ½\,(2 + 3\,i)$ since ceiling($|2 + 3\,i|$) = 4 is even.

Plainly, F is an extension of the Collatz function f, that is, $F(n) = f(n)$ for all n. Hence, the $3x + 1$ conjecture for f can be extended naturally to one for F: the trajectory of F at n eventually reaches 1. Equivalently, the trajectory of F at n ends in the cycle 1, 4, 2. Another way of saying this is that the trajectory of F at n can be partitioned into three subsequences a, b, c such that $a \to 1$, $b \to 4$, $c \to 2$.

But how does one generalize to trajectories of F at complex numbers z? Now, a little experimentation with different values of z will convince one that the above cyclic behavior need not be restricted to positive integers n. For example, if $z = 1 + i$, then the iterates $F^{192}(z)$, $F^{195}(z)$, …, $F^{200}(z)$ are approximately:

$$0.9999999966364368 + 1.009068983315935 \times 10^{-8}\, i$$

$$3.9999999899093104 + 3.027206949947805 \times 10^{-8}\, i$$

$$1.9999999949546552 + 1.5136034749739026 \times 10^{-8}\, i$$

$0.9999999974773276 + 7.568017374869513 \times 10^{-9}\, i$

$3.9999999924319827 + 2.2704052124608538 \times 10^{-8}\, i$

$1.9999999962159913 + 1.1352026062304269 \times 10^{-8}\, i$

$0.9999999981079957 + 5.6760130311521346 \times 10^{-9}\, i$

$3.999999994323987 + 1.7028039093456402 \times 10^{-8}\, i$

$1.9999999971619935 + 8.514019546728201 \times 10^{-9}\, i$

Observe that 1, 4, 2 appear to be cluster points of the trajectory of F at $1 + i$. This is indeed the case, as a later result will show. Hence, the trajectory at $1 + i$ can be partitioned into three subsequences a, b, c such that $a \to 1$, $b \to 4$, $c \to 2$, which is another way of stating the $3x + 1$ conjecture!

Call the following property of z the *tri-convergence property*: the trajectory of F at z can be partitioned into three subsequences a, b, c such that $a \to 1$, $b \to 4$, $c \to 2$.

<u>Theorem</u> If, for some positive integer R, the trajectory of F at z has consecutive terms

$$w_0 = F^R(z) = x_0 + y_0\, i,$$

$$w_1 = F^{R+1}(z) = x_1 + y_1\, i,$$

$$w_2 = F^{R+2}(z) = x_2 + y_2\, i,$$

with ceiling($|w_0|$) = 1, ceiling($|w_1|$) = 4, ceiling($|w_2|$) = 2, and $x_0 \sim 1$, $x_1 \sim 4$, $x_2 \sim 2$, then z satisfies the tri-convergence property. (For practical purposes, one can take $0.9 \leq x_0 \leq 1$, $3.9 \leq x_1 \leq 4$, $1.9 \leq x_2 \leq 2$.) In particular, the subsequences $a = \{F^S(z)\colon S = R \pmod 3\}$, $b = \{F^S(z)\colon S = R{+}1 \pmod 3\}$, $c = \{F^S(z)\colon S = R{+}2 \pmod 3\}$ converge to 1, 4, 2, respectively.

The proof appears at the end of this section. From the theorem and the values of the iterates $F^{192}(z)$, $F^{195}(z)$, $F^{196}(z)$ above, it is seen that $1 + i$ satisfies the tri-convergence property. Similarly, it can be shown that so do $1 + 2\,i$ and $7 + 3\,i$. This property is the key to extending the $3x + 1$ conjecture to complex numbers. ***The 3x + 1 conjecture can be restated as: the tri-convergence property holds for all n.*** To obtain a generalization, one looks for sets of complex numbers satisfying the property.

One need not look far to find complex numbers that probably do not satisfy the tri-convergence property. For instance, after 10^5 iterations of F, the initial value $z = 3 + 5\,i$ will have grown to about $1.25 \times 10^{12} + 1.42 \times 10^{12}\,i$. It is very unlikely that $3 + 5\,i$ satisfies the tri-convergence property.

For which regions in ***C*** can one conjecture that the tri-convergence property holds? This question will be addressed in the next section on graphics. The above theorem is now proved.

Sketch of Proof To prove convergence of the subsequences, assume first that if $w = F^r(z)$, $r = R, R + 1, R + 2, \ldots$, then ceiling($|w|$) cycles through 1, 4, 2. (This will be demonstrated later in this proof.)

Observe that, starting with any of the three iterates $w = w_i$ ($i = 0, 1, 2$) listed above, Im(w) is contracted by a factor of ¾ after going through a complete cycle consisting of three applications of F, hence can be made arbitrarily close to 0 by repeated cycling. Next, Re(w) is subjected by such a cycle to one of the transformations t_0: $x \rightarrow (3x + 1)/4$, t_1: $x \rightarrow \frac{3}{4}x + 1$, or t_2: $x \rightarrow \frac{3}{4}x + \frac{1}{2}$, according as ceiling($|w|$) = 1, 4, or 2, respectively. This implies that, by repeated cycling, Re(w) can be made arbitrarily close to 1, 4, or 2, according as ceiling($|w|$) = 1, 4, or 2, respectively. For example, if ceiling($|w|$) = 1, so that $x = \mathrm{Re}(w) \leq 1$, then $t_0(x) \leq 1$ and $|1 - t_0(x)| = ¾\,|1 - x|$, implying that the distance from $t_0(x)$ to 1 is ¾ the distance from x to 1. Therefore, $x < t_0(x)$ (unless $x = 1$ already, which is a trivial case) and $t_0^n(x) \rightarrow 1$. (In fact, $|1 - t_1(x)| = ¾\,|4 - x|$ and $|1 - t_2(x)| = ¾\,|2 - x|$, so the contraction factor ¾ is shared in common.)

From the foregoing remarks, it is clear that the subsequences $a = \{F^S(z)\text{: } S = R \text{ (mod 3)}\}$, $b = \{F^S(z)\text{: } S = R{+}1 \text{ (mod 3)}\}$, $c = \{F^S(z)\text{: } S = R{+}2 \text{ (mod 3)}\}$ converge to 1, 4, 2, respectively.

Now, to show that if $w = F^r(z)$, $r = R, R + 1, R + 2, \ldots$, then ceiling($|w|$) cycles through 1, 4, 2, consider $w = w_0$ and let $x = x_0$, $y = y_0$, so that one cycle brings x to $(3x + 1)/4$ and y to ¾ y. Since $|w| = x^2 + y^2 \leq 1$, it follows that $|F^{R+3}(z)| \leq 1$, because any point (x, y) that lies in the unit disc $x^2 + y^2 \leq 1$ also lies in the disc $|F^{R+3}(z)|^2 = [(3x + 1)/4]^2 + (¾\, y)^2 \leq 1$. (Completing the square reveals that this is the same disc as $(x + 1/3)^2 + y^2 \leq 16/9$.) But obviously, $|F^{R+3}(z)| > 0$, forcing ceiling($|F^{R+3}(z)|$) = 1. Similarly, by considering $w = w_1 = F^{R+1}(z)$ for which $|w| \leq 4$, it is seen that $3 < |F^{R+4}(z)|$ (this uses the hypothesis $x_0 \sim 1$, $x_1 \sim 4$, $x_2 \sim 2$) and $|F^{R+4}(z)| \leq 4$, and thus, ceiling($|F^{R+4}(z)|$) = 4. By the same argument, ceiling($|F^{R+5}(z)|$) = 2, and so on. This completes the sketch of proof.

3. Visualizing the 3z + 1 Dynamical System

To identify candidates that probably satisfy the tri-convergence property, it is useful to construct a density plot of F^N (the N-fold composition of F with itself) for a fixed large N. A *density plot* of the function g is a coloring of some region R of $\boldsymbol{C}$ where g is defined.

Typically, the larger the modulus of $g(z)$, the lighter the color assigned to the point $z \in \boldsymbol{C}$.

For simplicity, the density plots of F^N that will be considered here are over square regions R of **C**. A rectangular grid of P^2 squares is first superimposed on R; the number P is called the *resolution* of the density plot. The modulus of each $F^N(p)$, where p is the center of a grid square, is then computed, and the corresponding square is colored according to the size of the modulus.

Unless otherwise stated, the density plots in this paper have N (the number of F-iterations) = 400, and P (the resolution) = 400. The color code uses a simple monochrome rule:

Value of $m = \|F^N(p)\|$	Color
$0 \leq m < 5$	Black
$m \geq 5$	White

(Of course, more striking effects can be achieved using several colors.) Hence, regions in **C** that are colored black will represent regions in which the tri-convergence property probably holds.

4. Some Generalizations of the 3*x* + 1 Problem

The density plots of F^N over regions R centered at positive reals are predominantly black. This observation, combined with actual calculations of $|F^N(p)|$ for p lying on the real line $\text{Im}(z) = 0$, motivates the following conjectures.

I. The tri-convergence property holds for each real number > -½ (more precisely, for each z on the ray ρ defined by $\text{Re}(z) > -½$, $\text{Im}(z) = 0$).

II. For each z on the ray ρ, there is an open disk centered at z (that is, a set of the form $\{w: |w - z| < \varepsilon\}$ for some $\varepsilon > 0$) in which the tri-convergence property holds.

One can also consider a third statement that is perhaps less likely than the previous two above.

III. There is a semi-infinite strip σ defined by $\mathrm{Re}(z) > -½$, $|\mathrm{Im}(z)| < \varepsilon$, for some $\varepsilon > 0$, in which the tri-convergence property holds.

Each of these conjectures, being more expansive than the original $3x + 1$ problem, is probably at least as difficult to resolve.

5. The 3*x* + 1 Fractal and the Fang Motif

Figure 1 shows a density plot of F^N for R centered at the origin $0 + 0\,i$ and with edge length = 20. From this "bird's eye view" plot, it is readily observed that interesting things happen along the negative real line $\mathrm{Re}(z) < 0$, $\mathrm{Im}(z) = 0$.

Zooming in on the action, Figure 2 (a) illustrates the "fang motif" that appears to be repeated on an ever diminishing scale. Notice the sequence of fangs that seems to converge to the point $-2 + 0\,i$. This observation is reinforced by Figure 2 (b) which "magnifies" an even smaller square to the immediate right of $-2 + 0\,i$. The same pattern of fang motifs appears again!

Because of this self-similarity, the gray region is called the *3x + 1 fractal*.

Of course, the point $-2 + 0\,i$ is not unique. Many points along the negative real line are limit points of fang sequences. These points become more noticeable at greater magnifications. One example is the point -1.8 + 0 i (or some point very close to this). Figure 3 exhibits the fang sequence at -1.8 + 0 i.

Two problems for further research that immediately come to mind are:

(i) Characterize the limit points of fang sequences.

(ii) Estimate the fractal dimension of the $3x + 1$ fractal in regions containing limit points, such as those in Figure 3.

On a lighter note, the fang motif can be compared to the vaguely fractal creature in the "Alien" science fiction film series. This creature had great fangs, but more remarkably, it could lash out a tongue tipped with a miniature head having smaller fangs of its own.

6. Variants

Two variants of the complex Collatz function F are obtained by considering in the definition of F, instead of ceiling($|z|$), the functions floor($|z|$) = greatest integer $\leq |z|$ and round($|z|$) = integer nearest to $|z|$, rounding up in case of ambiguity. While these are interesting in their own right, the corresponding density plots do not have the lavish self-similarity exhibited using the standard version of F. Typical density plots with R centered at the origin are displayed in Figure 4.

Dedication This paper is dedicated to Dr. Mohammad R. Khadivi, a key influence in my interest in fractals.

References

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The Figures

The following figures were generated using the Java Abstract Windowing Toolkit (AWT) in the IBM Visual Age for Java development environment.

Figure 1. Bird's Eye View of the $3x + 1$ Fractal. Center: $0 + 0\,i$, edge length = 20.
Figure 2. The Fang Motif Repeated on an Ever Diminishing Scale at $-2 + 0\,i$.

a. Center: $0 + 0\,i$, edge length = 4.
b. Center: **:** $-1.75 + 0\,i$, edge length = 0.5.

Figure 3. The Fang Sequence at $-1.8 + 0\,i$. Center: $-1.55 + 0\,i$, edge length = 0.5.
Figure 4. Variants.

a. Using floor($|z|$). Center: $0 + 0\,i$, edge length = 4.
b. Using round($|z|$). Center: $0 + 0\,i$, edge length = 4.

Figure 1

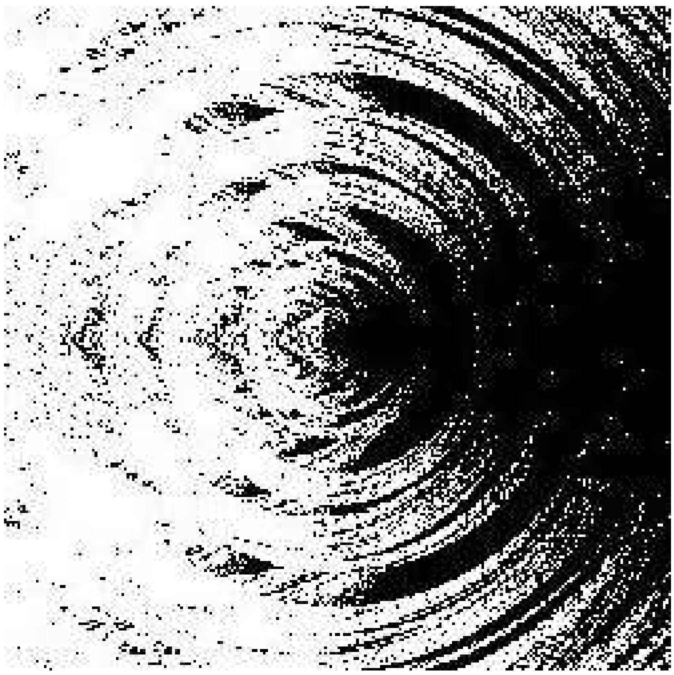
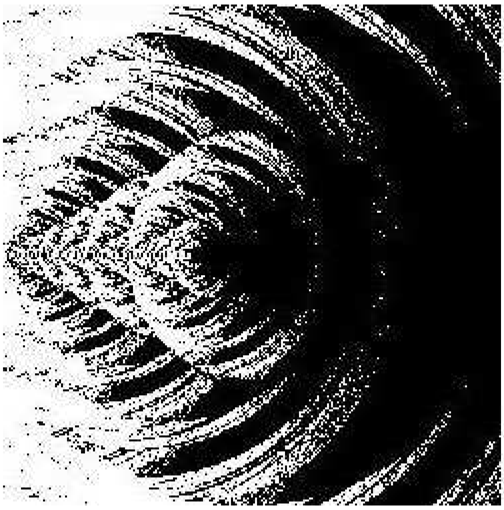


Figure 2


(a)

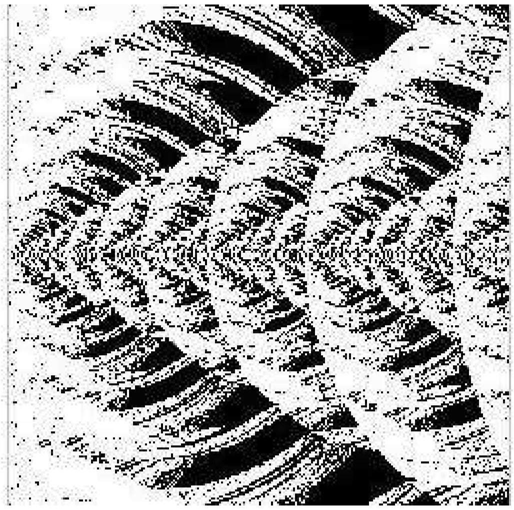

(b)

Figure 3

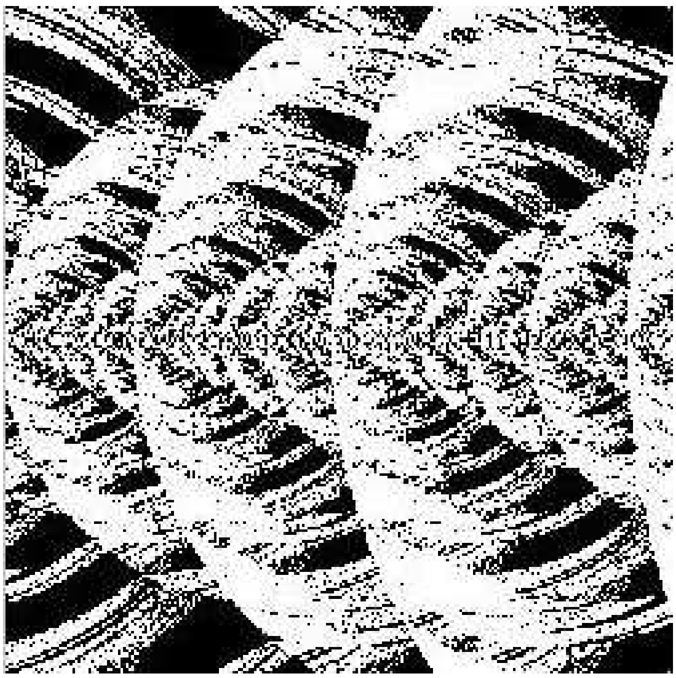
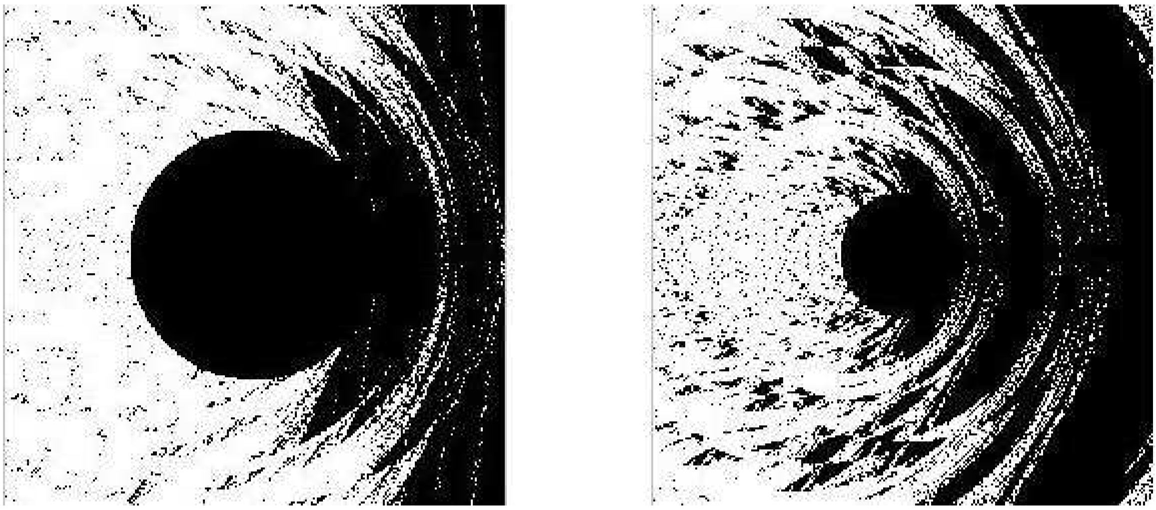


Figure 4



(a) (b)